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Finite-Volume Glauber Dynamics in a Small Magnetic Field

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We consider Glauber dynamics on a finite cube in the d-dimensional lattice ($d \geqslant 2$), which is associated with basic Ising model at temperature $T = 1/\beta \ll 1$ under a magnetic field $h > 0$. We prove that if the "effective magnetic field" is positive, then the relaxation of the Glauber dynamics in the uniform norm is exponentially fast, uniformly over the size of underlying cube. The result covers the case of the free-boundary condition with arbitrarily small positive magnetic field. This paper is a continuation of an attempt initiated earlier by Schonmann and Yoshida to shed more light on the relaxation of the finite-volume Glauber dynamics when the thermodynamic parameter (β, h) is so near the phase transition line, $\{(\beta, h); \beta_c < \beta \& h = 0\}$, that the Dobrushin–Shlosman mixing condition is no longer available.

KEY WORDS: Ising model; mixing conditions; Basuev region; boundary conditions; Glauber dynamics; exponential relaxation, spectral gap.

1. INTRODUCTION

We begin by reviewing the standard setup of the model. We then introduce some more specific backgrounds of our results.

The lattice. We will work on the d-dimensional integer lattice $\mathbf{Z}^d = \{x = (x^i)_{i=1}^d : x^i \in \mathbf{Z}\}$ with $d \geqslant 2$, on which we consider the l_1-metric; $\|x\|_1 = \sum_{i=1}^d |x^i|$. The number of points contained in a set $\Lambda \subset \mathbf{Z}^d$ is denoted

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by $|\Lambda|$ and we write $\Lambda \subset\subset \mathbf{Z}^d$ when $1 \leqslant |\Lambda| < \infty$. The interior and exterior boundaries of a set $\Lambda \subset \mathbf{Z}^d$ will be denoted respectively by

$$\partial_{\text{int}} \Lambda = \{x \in \Lambda;\ \|x-y\|_1 = 1 \text{ for some } y \notin \Lambda\},$$

$$\partial_{\text{ext}} \Lambda = \{x \notin \Lambda;\ \|x-y\|_1 = 1 \text{ for some } y \in \Lambda\},$$

A set $\Lambda \subset\subset \mathbf{Z}^d$ is called a *box* when $\Lambda = \mathbf{Z}^d \cap \prod_{j=1}^d [a^j, b^j)$ for some integers $a^j < b^j$ $(j=1,..., n)$. A box Λ, expressed as above is called a *cube* when all side-lengths $b^j - \alpha^j$ $(j=1,..., n)$ are identical. For $x \in \mathbf{Z}^d$ and $l = 1, 2,...$, a cube with the center x and the side-length $2l$ is denoted by $Q_x(l)$; $Q_x(l) = \mathbf{Z}^d \cap \prod_{i=1}^d [x^i - l, x^i + l)$. $Q_0(l)$ will be simply denoted by $Q(l)$.

The configurations and the Gibbs states. Configuration spaces are defined as follows;

$$S^\Lambda = \{\sigma = (\sigma_x)_{x \in \Lambda};\ \sigma_x \in \{-1, +1\}\},\ \Lambda \subset\subset \mathbf{Z}^d,$$

$$\Omega = \{\omega = (\omega_x)_{x \in \mathbf{Z}^d};\ \omega_x \in \{-1, 0, +1\}\}.$$

We will refer an element ω of Ω as a boundary condition. Three boundary conditions are specially relevant: $(\pm)$ and (0), which are defined respectively by $(\pm)_x \equiv \pm 1$ and $(0)_x \equiv 0$. $(\pm)$ is called pure $(\pm)$-boundary condition and (0) is called the free-boundary condition. They will often be abbreviated respectively by $\pm$ and 0 especially when they appear as superscript. For $\Lambda \subset\subset \mathbf{Z}^d$ and $(\sigma, \omega) \in S^\Lambda \times \Omega$, $\sigma_\Lambda \cdot \omega_{\Lambda^c}$ denotes the following configuration:

$$(\sigma_\Lambda \cdot \omega_{\Lambda^c})_x = \begin{cases} \sigma_x & \text{if } x \in \Lambda \\ \omega_x & \text{if } x \notin \Lambda \end{cases}$$

The set of all real function on S^Λ is denoted by $\mathscr{C}_\Lambda$. For $\Lambda \subset\subset \mathbf{Z}^d$ and $\omega \in \Omega$, *Hamiltonian* $H^{\Lambda, \omega} \in \mathscr{C}_\Lambda$ is defined by

$$H^{\Lambda, \omega}(\sigma) = -\sum_{\substack{\{x, y\} \subset \Lambda \\ \|x-y\|_1 = 1}} \beta \sigma_x \sigma_y - \sum_{x \in \Lambda} \beta \sigma_x \left\{h + \sum_{\substack{y \notin \Lambda \\ \|x-y\|_1 = 1}} \omega_y\right\} \tag{1.1}$$

where $\beta > 0$ and $h \geqslant 0$ denote respectively the inverse temperature and the magnetic field. The real numbers;

$$\left(h + \sum_{\substack{y \notin \Lambda \\ \|x-y\|_1 = 1}} \omega_y\right)_{x \in \Lambda} \tag{1.2}$$

which appear in the second summation on the right-hand-side of (1.1) is often called the effective magnetic field. *Gibbs state* on Λ with the boundary condition $\omega \in \Omega$ is defined as a probability distribution $\mu^{\Lambda, \omega} = \mu^{\Lambda, \omega}_{\beta, h}$ on S^Λ, in which the probability of each configuration $\sigma \in S^\Lambda$ is given by

$$\mu^{\Lambda, \omega}\{\sigma\} = \frac{1}{Z^{\Lambda, \omega}} \exp -H^{\Lambda, \omega}(\sigma) \tag{1.3}$$

where $Z^{\Lambda, \omega}$ is the normalizing constant. We will use the following common and convenient abbreviation; $\mu^{\Lambda, \omega} f = \sum_{\sigma \in S^\Lambda} \mu^{\Lambda, \omega}\{\sigma\} f(\sigma)$ for $f \in \mathscr{C}_\Lambda$. The following facts are well known. (i) The limit $m^{\pm}_{\beta, h} = \lim_{\Lambda \uparrow \mathbf{Z}^d} \mu^{\Lambda, \pm}_{\beta, h}(\sigma_0)$ exists. (ii) $m^+_{\beta, h} = m^-_{\beta, h} > 0$ if $h > 0$. (iii) There exists a *critical inverse temperature* $\beta_c \in (0, \infty)$ such that $m^+_{\beta, 0} = m^-_{\beta, 0} = 0$ if $\beta < \beta_c$ and $m^+_{\beta, 0} > 0 > m^-_{\beta, 0}$ if $\beta_c < \beta$. Therefore, the (β, h)-plane is divided into two parts according as $m^+_{\beta, h} = m^-_{\beta, h}$ or not, up to the critical point $(\beta_c, 0)$;

$$\{(\beta, h); \beta < \beta_c \text{ or } h > 0\}, \qquad \{(\beta, h); \beta_c < \beta \,\&\, h = 0\} \tag{1.4}$$

The former is called *one phase region* and the latter is called *phase transition line.*

The Glauber dynamics. For $f \in \mathscr{C}_\Lambda$, we introduce a difference operator

$$\nabla_x f(\sigma) = f(\sigma^x) - f(\sigma), \qquad x \in \Lambda$$

where σ^x is defined by;

$$\sigma^x_y = \begin{cases} -\sigma_y & \text{if } \; y = x \\ \sigma_y & \text{if } \; y \neq x \end{cases}$$

We also introduce a uniform norm; $\|f\| = \sup_\sigma |f(\sigma)|$ and a semi-norm; $|||f||| = \sum_{x \in \Lambda} \|\nabla_x f\|$. Now, fix $\Lambda \subset\subset \mathbf{Z}^d$ and $\omega \in \Omega$. We define a linear operator $A^{\Lambda, \omega}: \mathscr{C}_\Lambda \to \mathscr{C}_\Lambda$ by

$$A^{\Lambda, \omega} f(\sigma) = \sum_{x \in \Lambda} c^{\Lambda, \omega}_x(\sigma) \nabla_x f(\sigma), \qquad f \in \mathscr{C}_\Lambda \tag{1.5}$$

where

$$c^{\Lambda, \omega}_x(\sigma) = \exp -\nabla_x H^{\Lambda, \omega}(\sigma) \tag{1.6}$$

It can easily be seen that

$$-\mu^{\Lambda, \omega}(f A^{\Lambda, \omega} g) = \frac{1}{2} \sum_{x \in \Lambda} \mu^{\Lambda, \omega}(c^{\Lambda, \omega}_x \nabla_x f \; \nabla_x g)$$

which shows that $A^{\Lambda,\omega}$ is $\mu^{\Lambda,\omega}$-symmetric and non-positive definite. Finally, we define

$$\mathrm{gap}_{\beta,h}^{\Lambda,\omega} = \inf\{\lambda > 0;\ \mathrm{Ker}(\lambda - (-A^{\Lambda,\omega})) \neq \{0\}\} > 0 \tag{1.7}$$

which is called the *spectral gap*. $\mathrm{gap}_{\beta,h}^{\Lambda,\omega}$ can be characterized as the largest number that satisfies

$$\| T_t^{\Lambda,\omega} f - \mu^{\Lambda,\omega} f \|_{L^2(\mu^{\Lambda,\omega})} \leqslant \| f - \mu^{\Lambda,\omega} f \|_{L^2(\mu^{\Lambda,\omega})} \exp(-t\, \mathrm{gap}_{\beta,h}^{\Lambda,\omega}) \tag{1.8}$$

for all $f \in \mathscr{C}_\Lambda$, where

$$T_t^{\Lambda,\omega} = \exp tA^{\Lambda,\omega}, \qquad t > 0 \tag{1.9}$$

Some backgrounds of our result. For given $\beta > 0$, $h \in \mathbf{R}$ and $\omega \in \Omega$, we are interested in whether the following exponential relaxation property in a uniform sense is true;

there exist constants $C_i = C_i(d, \beta, h, \omega) \in (0, \infty)$ $(i = 1, 2)$ such that for all cube Λ, $f \in \mathscr{C}_\Lambda$ and $t > 0$

$$\| T_t^{\Lambda,\omega} f - \mu^{\Lambda,\omega} f \| \leqslant C_1 \||f\|| \exp -\frac{t}{C_2} \tag{1.10}$$

Note first of all that the above exponential relaxation property can be true only in the one phase region. In fact, it is not difficult to see that, if (1.10) holds for some β, h and $\omega \in \Omega$, then $m_{\beta,h}^+ = m_{\beta,h}^-$. Here we summarize some of known results and a conjecture;

(i) The exponential relaxation property (1.10) is true for all $\omega \in \{-1, +1\}^{\mathbf{Z}^d}$, whenever a mixing condition in the sense of the Dobrushin–Shlosman in a certain restricted sense is valid ([MO94a, MO94b], See also [SZ92]). This is the case if $d = 2$ and (β, h) is in the one phase region ([SS95]) or $d \geqslant 3$ and (β, h) is sufficiently deep inside the one phase region ([MO94a])

(ii) It is conjectured that if $d \geqslant 3$, β is sufficiently large and h is a certain small positive number depending on β, then there is an $\omega \in \Omega$ which violates the exponential relaxation property. The part of the (β, h)-plane reffered to above is often called "Basuev region."

Though the conjecture (ii) above is not vindicated yet, there are some results on related models which suggest the existence of such "dangerous" boundary condition $\omega \in \Omega$ ([CM96a, CM96b, DM94]). On the other hand, the conjecture motivated an attempt to find out a class of "safe" boundary conditions when (β, h) is sitting in the one phase region but near the phase transition line. In fact, it is proved in [SY97, Theorem 4.1] that the exponential relaxation property (1.10) is true (for any $\Lambda \subset\subset \mathbf{Z}^d$, not only for cubes) at least in the following three cases: (a) $d \geqslant 2$, $\beta < \beta_c$, $h \geqslant 0$ and $\omega \in \Omega$ is such that all components of the effective magnetic field (cf. (1.2)) are non-negative for any Λ, (b) $d = 2$, $\beta \geqslant \beta_c$, $h > 0$ and $\omega \equiv +1$, (c) $d \geqslant 3$, β is sufficiently large, $h > 0$ and $\omega \equiv +1$. In the present paper, we continue the program in [SY97] to understand more about the exponential relaxation property in the Basuev region. A technical constraint in working in the region where the Dobrushin–Shlosman mixing condition is no longer available is that we have to abandon the use of logarithmic Sobolev inequality, which is in a very powerful analytic tool to establish the exponential relaxation property ([MO94b, SZ92]). We instead use the more specific structures of the ferromagnetic Ising model, especially attractivity and Ising percolation.

2. RESULTS

The main result in this paper is the following

Theorem 2.1. There exists $\beta_0 = \beta_0(d) \in (0, \infty)$ for which the following hold.

(a) For any $\beta > \beta_0$ and $h_0 > 0$, there exist constants $C_i = C_i(d, \beta, h_0) \in (0, \infty)$ $(i = 1, 2)$ such that if $h > 0$, $\omega \in \Omega$ and a cube Λ satisfy

$$\min_{x \in \Lambda} \Bigg\{ h + \sum_{\substack{y \notin \Lambda \\ \|x-y\|_1 = 1}} \omega_y \Bigg\} \geqslant h_0 \tag{2.1}$$

then for all $f \in \mathscr{C}_\Lambda$ and $t > 0$

$$\| T_t^{\Lambda, \omega} f - \mu^{\Lambda, \omega} f \| \leqslant C_1 \, |||f||| \exp -\frac{t}{C_2} \tag{2.2}$$

(b) For $\beta > \beta_0$,

$$0 < \inf\{ \mathrm{gap}_{\beta, h}^{\Lambda, \omega};\ h > 0,\ \omega \in \Omega \text{ and a cube } \Lambda \text{ satisfy (2.1)} \} \tag{2.3}$$

Furthermore, in the case of the free-boundary condition, there exist $C_3 = C_3(d, \beta) \in (0, \infty)$ and $C_4 = C_4(d) \in (0, \infty)$ such that for any $h \in (0, 1)$

$$0 < \inf\{ \mathrm{gap}_{\beta, h}^{\Lambda, 0};\ \Lambda \text{ is a cube} \} \leqslant C_3 \exp -\frac{\beta}{C_4 h^{d-1}} \tag{2.4}$$

Remark 2.1. The condition (2.1) is satisfied for any $h > 0$, if all ω_y's are non-negative. It is the case of very small h when the conclusion of the theorem becomes really interesting because, for $d \geqslant 3$, it covers the Basuev region reffered to in the previous section. Even for $d = 2$, our result seems to be new, since it also applies to the free boundary condition. Part(a) for pure $(+)$-boundary condition has already been obtained in [SY97, Theorem 4.1, part(b)] (in fact for any $\Lambda \subset\subset \mathbf{Z}^d$, not only for cubes). The extension to the present case, which includes the case of free boundary condition with any positive magnetic field was a question left unsolved in that paper. The uniform positivity of the spectral gap in (2.4) is in sharp contrast with what happens at the phase transition line of $h = 0$. In fact, [T89, Proposition 2.4] says that there are $C_i = C_i(d) \in (0, \infty)$ $(i = 1, 2)$ such that for sufficiently large β and $h = 0$,

$$\text{gap}_{\beta, 0}^{Q(l), 0} \leqslant \exp(\beta C_1 - \beta C_2 l^{d-1}) \tag{2.5}$$

for any $l \geqslant 1$. On the other hand, the right-hand-side estimate in (2.4) is a manifestation of the phase-coexistence at $h = 0$. This contrasts with the situation at high temperature; [SY97, Theorem 4.1, part(a)] implies in particular that $0 < \inf\{\text{gap}_{\beta, h}^{\Lambda, 0};\ \Lambda \subset\subset \mathbf{Z}^d,\ h \geqslant 0\}$ if $\beta < \beta_c$.

Part (a) of Theorem 2.1 is obtained by combination of Proposition 2.2 and Proposition 2.3, which we present below.

Proposition 2.2 [SY972]. Suppose that $\Lambda \subset\subset \mathbf{Z}^d$ and that $\{B_1, B_2\} \subset (0, \infty)$ are such that for a certain boundary condition $\omega \in \Omega$

$$\sup_{x \in \Lambda} \{\mu^{\Lambda \cup Q_x(l), (+)_\Lambda \cdot \omega_{\Lambda^c}}(\sigma_x) - \mu^{\Lambda \cap Q_x(l), (-)_\Lambda \cdot \omega_{\Lambda^c}}(\sigma_x)\}$$

$$\leqslant B_1 \exp -\frac{l}{B_2}, \qquad l = 1, 2, \ldots \tag{2.6}$$

Then there exists $\{C_1, C_2\} \subset (0, \infty)$ which depend only on d, β, h, B_1 and B_2 above such that (2.2) holds for all $f \in \mathscr{C}_\Lambda$ and all $t > 0$.

Proposition 2.3. There exists $\beta_0 = \beta_0(d) \in (0, \infty)$ for which the following holds. For any $\beta \geqslant \beta_0$ and $h_0 \in (0, \infty)$, there exist constants $B_i = B_i(d, \beta, h_0) \in (0, \infty)$ $(i = 1, 2)$ such that (2.6) holds for all $\beta \geqslant \beta_0$, $h > 0$, $\omega \in \Omega$ and a cube Λ which satisfy (2.1).

In Section 3, we will prove Proposition 2.3 by using Proposition 2.4 below, which is an extension of [Sch94a, Theorem 3]. To state Proposition 2.4, we need to introduce some more notations. We take a subset

$E \subset \{\pm 1, ..., \pm d\}$ to specify some of $2d$ basic directions in the lattice. We suppose that E is non-degenerate;

$$E \cap \{-j, j\} \neq \phi, \qquad \forall j = 1, ..., d \tag{2.7}$$

We first describe a boundary condition around a box Λ, which equals $(\pm)$ on faces in E-directions and equals (0) on the other faces. Let $\delta_j = (\delta_{jk})_{k=1}^d \in \mathbf{Z}^d$ and $\delta_{-j} = -\delta_j$ $(j = 1, 2, ..., d)$. For a box $\Lambda \subset \mathbf{Z}^d$, $\partial_{\text{ext}} \Lambda$ is the disjoint union of $\partial_{\text{ext}}^{(j)} \Lambda$ for $j = \pm 1, ..., \pm d$, where

$$\partial_{\text{ext}}^{(j)} \Lambda = \{ y \in \partial_{\text{ext}} \Lambda;\ y - \delta_j \in \Lambda \}$$

We let $(\pm; E)$ denote a configuration on $\partial_{\text{ext}} \Lambda$ which is $\equiv \pm 1$ on $\bigcup_{j \in E} \partial_{\text{ext}} \Lambda$ and $\equiv 0$ on $\bigcup_{j \notin E} \partial_{\text{ext}}^{(j)} \Lambda$.

We next define a box $Q(l; E)$ $(l > 0)$ by

$$Q(l; E) = \prod_{j=1}^{d} Q^{(j)}(l) \tag{2.8}$$

where

$$Q^{(j)}(l) = \begin{cases} \mathbf{Z} \cap [-l, l) & \text{if } \ \pm j \in E \\ \mathbf{Z} \cap [0, l) & \text{if } \ -j \notin E \ \text{ and } \ j \in E \\ \mathbf{Z} \cap [-l, 0) & \text{if } \ -j \in E \ \text{ and } \ j \notin E \end{cases}$$

As l increases, $Q(l; E)$ grows larger only in the directions specified by E.

Finally for $A \subset \Lambda \subset \mathbf{Z}^d$, $\sigma \in S^\Lambda$ and $\varepsilon = +$ or $-$, (ε)-*path* in A at a configuration σ is a sequence $\gamma = (x_i \in A)_{i=0}^n$ such that $\|x_i - x_{i-1}\|_1 = 1$ for $1 \leqslant i \leqslant n$ and $\sigma_x \equiv \varepsilon 1$ for $x \in \gamma$.

Proposition 2.4. For each $B \in (d, \infty)$ and $A \in (0, B)$, there exists $\beta_0 = \beta_0(d, A, B) \in (0, \infty)$ such that for $\beta \geqslant \beta_0$,

$$\overline{\lim_{h \searrow 0}}\, h \log \mu_{\beta, h}^{Q(B/h;\, E),\, (-;\, E)} \left(\mathscr{B}\left(\frac{A}{h}; E \right) \right) < 0 \tag{2.9}$$

where $\mathscr{B}(A/h; E)$ stands for an event that there is a $(-)$-path in $Q(B/h; E)$ which connects $Q(A/h; E)$ to a site adjacent to $\bigcup_{j \in E} \partial_{\text{ext}}^{(j)} Q(B/h; E)$.

Remark 2.2. In the case of $E = \{\pm 1, ..., \pm d\}$, in which the smaller box $Q(A/h; E)$ goes far away from $\partial_{\text{ext}}\, Q(B/h; E)$ as $h \searrow 0$. Proposition 2.4 is what can be read off directly from the proof of [Sch94a, Theorem 3]. We also need to consider the case of $E \neq \{\pm 1, ..., \pm d\}$, in which the situation is slightly different, since as $h \searrow 0$, the smaller box $Q(A/h; E)$ goes far away from $\bigcup_{j\in E} \partial_{\text{ext}}^{(j)} Q(B/h; E)$ where $(-)$-boundary condition is put, but it stays adjacent to $\bigcup_{j\notin E} \partial_{\text{ext}}^{(j)} Q(B/h; E)$ where free-boundary condition is put. The upshot is that it is possible to modify the argument in the proof of [Sch94a, Theorem 3] to cover the case of $E \neq \{\pm 1,..., \pm d\}$, as far as (2.7) is satisfied. The proof of Proposition 2.4 will be sketched in Section 4.

For the part (b) of Theorem 2.1, the uniform positivity of the spectral gap in both (2.3) and (2.4) is the consequence of the part (a). The right-hand-side estimate of (2.4) comes from the following

Proposition 2.5. There exist $\beta_0 = \beta_0(d) \in (0, \infty)$ and $C_1 = C_1(d) \in (0, \infty)$ for which the following holds. For any $\beta \geqslant \beta_0$ and $B \geqslant 1$ there exists $C_2 = C_2(d, \beta, B) \in (0, \infty)$ such that for any $h \in (0, 1)$,

$$\text{gap}_{\beta, h}^{Q(B/h), 0} \leqslant C_2 \exp -\frac{\beta}{C_1 h^{d-1}} \tag{2.10}$$

Remark 2.3. It is not difficult to see that

$$\text{gap}_{\beta, h}^{Q(B/h), 0} \geqslant \frac{1}{C(d, \beta, B)} \exp -\frac{C(d, \beta, B)}{h^{d-1}}$$

for some $C(d, \beta, B) \in (0, \infty)$ ([Sch94a, Theorem 5], for example). Thus, the inequality (2.10) is sharp up to constants. Similar upper and lower bound of the spectral gap for $(-)$-boundary condition can be found in [Sch94b, Theorem 4]

Remark 2.4. Proposition 2.5 says a little more than is needed to conclude the right-hand-side estimate of (2.4). In fact, the right-hand-side estimate of (2.4) can be obtained just by proving (2.10) for small enough B. The proof of (2.10) in this case is easy as we present now. It is not difficult to see that

$$\exp(-2h(B/h)^d) \leqslant \frac{\mu_{\beta, h}^{Q(B/h), 0}\{\sigma\}}{\mu_{\beta, 0}^{Q(B/h), 0}\{\sigma\}} \leqslant \exp(2\beta h(B/h)^d)$$

From this and a standard comparison argument, we get

$$\text{gap}_{\beta, h}^{Q(B/h), 0} \leqslant \text{gap}_{\beta, 0}^{Q(B/h), 0} \exp(\beta C_0 + 4\beta h(B/h)^d)$$

for some $c_0 = C_0(d) \in (0, \infty)$. We then apply (2.5) to see that the right hand side of the above inequality is bounded from above by

$$\exp\{\beta(C_0 + C_1) - \beta(B/h)^{d-1}(C_2 - 4B)\}$$

which proves (2.10) when $B < C_2/4$.

3. PROOF OF PROPOSITION 2.3

In this section, we give a proof of Proposition 2.3. We begin by reducing the proof to the case of $\omega \equiv 0$. This is possible by the following inequality;

$$\mu_{\beta, h}^{\Lambda \cap Q_x(l), (+)_\Lambda \cdot \omega_{\Lambda^c}}(\sigma_x) - \mu_{\beta, h}^{\Lambda \cap Q_x(l), (-)_\Lambda \cdot \omega_{\Lambda^c}}(\sigma_x) \leqslant \mu_{\beta, h_0}^{\Lambda \cap Q_x(l), (+)_\Lambda \cdot \omega_{\Lambda^c}}(\sigma_x) - \mu_{\beta, h_0}^{\Lambda \cap Q_x(l), (-)_\Lambda \cdot \omega_{\Lambda^c}}(\sigma_x) \tag{3.1}$$

The above inequality is proved as follows. This is where the condition (2.1) is used. For an inhomogeneous magnetic field $\{h_z \in \mathbf{R}; z \in \Lambda \cap Q_x(l)\}$, we define a probability measure $\nu^{\{h_z\}}$ on $S^{\Lambda \cap Q_x(l)}$ by

$$\nu^{\{h_z\}}\{\sigma\} = \frac{\exp -H_{\{h_z\}}(\sigma)}{\text{normalization}}$$

where

$$-H_{\{h_z\}}(\sigma) = \beta \sum_{\langle v, w \rangle} \sigma_v \sigma_w + \beta \sum_{z \in \Lambda \cap Q_x(l)} h_z \sigma_z$$

with $\sum_{\langle v, w \rangle}$ denoting the summation over all nearest neighbour pairs in the set $\Lambda \cap Q_x(l)$. We will need the following fact; if inhomogeneous magnetic fields $\{h_z^\pm\}$ and $\{k_z^\pm\}$ are given such that for each $z \in \Lambda \cap Q_x(l)$,

$$0 \leqslant h_z^+ - h_z^- \leqslant k_z^+ - k_z^- \tag{3.2}$$

and

$$0 \leqslant k_z^+ - k_z^- \leqslant h_z^+ - h_z^- \tag{3.3}$$

then

$$\nu^{\{h_z^+\}}(\sigma_x) - \nu^{\{h_z^-\}}(\sigma_x) \leqslant \nu^{\{k_z^+\}}(\sigma_x) - \nu^{\{k_z^-\}}(\sigma_x) \tag{3.4}$$

The proof of this fact, can be given in the same way as that of [Hig93, (2.15)]. At this point, we make the following particular choice of $\{h_z^\pm\}$ and $\{k_z^\pm\}$;

$$h_z^\pm = h + \sum_{\substack{y \notin \Lambda \cap Q_x(l) \\ \|y-z\|_1 = 1}} ((\pm)_\Lambda \cdot \omega_{\Lambda^c})_y$$

$$k_z^\pm = h_0 + \sum_{\substack{y \notin \Lambda \cap Q_x(l) \\ \|y-z\|_1 = 1}} ((\pm)_\Lambda \cdot 0_{\Lambda^c})_y$$

In this case (3.2) is obvious and (3.3) follows from (2.1). Since $\nu^{\{h_z^\pm\}} = \mu_{\beta, h}^{\Lambda \cap Q_x(l), (\pm)_\Lambda \cdot \omega_{\Lambda^c}}$ and $\nu^{\{k_z^\pm\}} = \mu_{\beta, h_0}^{\Lambda \cap Q_x(l), (\pm)_\Lambda \cdot \omega_{\Lambda^c}}$, (3.4) implies (3.1).

The basic strategy to prove that the right-hand-side of (3.1) decays exponentially in l is similar to Appendix 1 of [MO94a], where the "effectiveness" of a mixing condition for Gibbs states is discussed. We apply coupling technique to derive a recursive inequality, which leads to desired exponential decay. An alternative proof, which does not rely on such coupling technique is presented in [Y97].

Let us abbreviate $\Lambda \cap Q_x(l)$ and $(\pm)_\Lambda \cdot 0_{\Lambda^c}$ respectively by Γ and $\zeta^\pm$ for simplicity. It is convenient to introduce the following restricted configuration space; $S^{\Delta, \eta} = \{\sigma_\Delta \cdot \eta_{\Delta^c}; \sigma \in S^\Delta\}$ $(\Delta \subset\subset \mathbf{Z}^d, \eta \in \Omega)$. Define;

$$F_z(l) = \mu_{\beta, h_0}^{\Gamma, \zeta^+}\{\sigma_z = +1\} - \mu_{\beta, h_0}^{\Gamma, \zeta^-}\{\sigma_z = +1\}, \qquad z \in \Gamma$$

We are going to prove that $F_x(l)$ decays exponentially in l. To this end, we may and will assume that

$$(d+1)\,m + 1 < \frac{l}{2} \tag{3.5}$$

where $m = m(h, d, \beta) \geqslant 1/h$ is a large enough integer which will be specified at the end of the proof. It is enough to establish the following recursive inequality;

$$F_z(l) \leqslant e^{-1} \max\{F_{z'}(l); z' \in \Gamma \cap Q_z((d+1)\,m+1)\}, \qquad \text{if } z \in Q_x\left(\frac{l}{2}\right) \tag{3.6}$$

In fact, starting from $z = x$, we can iterate (3.6) at least $n = \lfloor (l/2)/m(d+1)+1 \rfloor$ times to obtain

$$F_x(l) \leqslant e^{-n} \max\{F_{z'}(l); z' \in \Gamma \cap Q_x(n(d+1)\,m+n)\} \leqslant e^{-n}$$

which is what we want to prove.

To prove (3.6), we begin with some geometry. For $z \in \Gamma$, define $E(z) = \{-2d \leqslant j \leqslant 2d;\ z + \delta_j \in \Gamma\}$. If $z \notin \partial_{\text{int}} \Gamma$, then $E(z) = \{\pm 1, \ldots, \pm d\}$. If $z \in \partial_{\text{int}} \Gamma$, then $E(z)$ is a collection of "inward" directions to Γ. Note that $E(z) \cap \{-j, +j\} \neq \phi$ for any $z \in \Gamma$. Furthermore, for each $z \in \Gamma$, we can find $\bar{z} \in \Gamma$ such that

$$z \in \bar{z} + Q(dm, E(\bar{z})) \subset \bar{z} + Q((d+1)\,m, \quad E(\bar{z})) \subset \Gamma \tag{3.7}$$

If z is near $\partial_{\text{int}} \Gamma$, we have to choose $\bar{z}$ from $\partial_{\text{int}} \Gamma$. We set $\Gamma(z) = \bar{z} + Q((d+1)\,m, E(\bar{z}))$. At this point, let us make the following observation for later use;

$$\{y \in \partial_{\text{ext}} \Gamma(z);\ \eta^1_y \neq \eta^2_y\} \subset \Gamma \cap \partial_{\text{ext}} \Gamma(z) = \bigcup_{j \in E(z)} \partial^{(j)}_{\text{ext}} \Gamma(z)$$

$$\text{if} \quad z \in Q_x\left(\frac{l}{2}\right), \quad \eta^1 \in S^{\Gamma, \zeta^-} \quad \text{and} \quad \eta^2 \in S^{\Gamma, \zeta^+} \tag{3.8}$$

This can be seen as follows. Since $z \in Q_x(l/2)$, we see from (3.5) that $\partial_{\text{ext}} \Gamma(z) \subset Q_x(l)$. This implies that $\partial_{\text{ext}} \Gamma(z) \backslash \Gamma$ is contained in $Q_x(l) \backslash \Gamma$, where both η^1 and η^2 are identical to 0.

We will use the following coupling (cf. [BM94, Theorem 2]); for each $\Delta \subset \Gamma$ and $(\eta^1, \eta^2) \in S^{\Gamma, \zeta^-} \times S^{\Gamma, \zeta^+}$, there is a probability measure $\mu^{\Delta, \eta^1, \eta^2}$ on the coupled configuration space $S^{\Delta, \eta^1} \times S^{\Delta, \eta^2} = \{(\sigma^1, \sigma^2);\ \sigma^i \in S^{\Delta, \eta^i}\}$ such that (i) $\mu^{\Delta, \eta^1, \eta^2}$ is a coupling of $\mu^{\Delta, \eta^1}_{\beta, h_0}$ and $\mu^{\Delta, \eta^2}_{\beta, h_0}$, i.e., its first and second marginals are $\mu^{\Delta, \eta^1}_{\beta, h_0}$ and $\mu^{\Delta, \eta^2}_{\beta, h_0}$, respectively. (ii) If $\eta^1 \leqslant \eta^2$, then $\mu^{\Delta, \eta^1, \eta^2}$ is above diagonal in that $\mu^{\Delta, \eta^1, \eta^2}\{\sigma^1 \leqslant \sigma^2\} = 1$. (iii) For $z \in \Delta$,

$$\mu^{\Delta, \eta^1, \eta^2}\{\sigma^1_z \neq \sigma^2_z\} = \mu^{\Delta, \eta^1, \eta^2} \left\{ \begin{array}{l} \text{There is a path } \gamma \text{ from } z \text{ to a site adjacent} \\ \text{to the set } \{y \in \partial_{\text{ext}} \Delta;\ \eta^1_y \neq \eta^2_y\} \text{ such that} \\ \sigma^1 \equiv -1 \text{ and } \sigma^2 \equiv +1 \text{ on } \gamma. \end{array} \right\} \tag{3.9}$$

We now claim that for $z \in Q_x(l/2)$ and $(\eta^1, \eta^2) \in S^{\Gamma, \zeta^-} \times S^{\Gamma, \zeta^+}$,

$$\mu^{\Gamma(z), \eta^1, \eta^2}\{\sigma^1_z \neq \sigma^2_z\} \begin{cases} = 0 & \text{if} \quad \eta^1 \equiv \eta^2 \text{ on } \Gamma \cap \partial_{\text{ext}} \Gamma(z), \\ \leqslant \exp(C_1 - m/C_1) & \text{otherwise} \end{cases} \tag{3.10}$$

where $C_1 = C_1(\beta, d) \in (0, \infty)$ is a constant. This can be seen as follows. If $\eta^1 \equiv \eta^2$ on $\Gamma \cap \partial_{\text{ext}} \Gamma(z)$, then it follows from (3.8) that $\eta^1 \equiv \eta^2$ on $\partial_{\text{ext}} \Gamma(z)$, which implies $\mu^{\Gamma(z), \eta^1, \eta^2}\{\sigma^1_z \neq \sigma^2_z\} = 0$ by (3.9). If $\eta^1 \not\equiv \eta^2$ on $\Gamma \cap \partial_{\text{ext}} \Gamma(z)$, we can use (3.8) and (3.9) to see that

$$\begin{aligned} &\mu^{\Gamma(z), \eta^1, \eta^2}\{\sigma^1_z \neq \sigma^2_z\} \\ &\quad \leqslant \mu^{\Gamma(z), \eta^1, \eta^2} \left\{ \begin{array}{l} \text{There is a path } \gamma \text{ from } z \text{ to a site adjacent} \\ \text{to the set } \Gamma \cap \partial_{\text{ext}} \Gamma(z) \text{ such that} \\ \sigma^1 \equiv -1 \text{ and } \sigma^2 \equiv +1 \text{ on } \gamma. \end{array} \right\} \\ &\quad \leqslant \mu^{\Gamma(z), \eta^1}_{\beta, h} \left\{ \begin{array}{l} \text{There is a } (-)\text{-path } \gamma \text{ from } z \text{ to a site} \\ \text{adjacent to the set } \Gamma \cap \partial_{\text{ext}} \Gamma(z) \end{array} \right\} \\ &\quad \leqslant \mu^{\Gamma(z), (-;\, E(z))}_{\beta, 1/m} \left\{ \begin{array}{l} \text{There is a } (-)\text{-path } \gamma \text{ from } z \text{ to a site} \\ \text{adjacent to the set } \Gamma \cap \partial_{\text{ext}} \Gamma(z) \end{array} \right\} \end{aligned} \tag{3.11}$$

where we have used the FKG inequality in passage to the last line. By (3.7) and Proposition 2.4, the probability appearing in (3.11) is bounded from above by $\exp(C_1 - m/C_1)$ with some $C_1 = C_1(d, \beta) \in (0, \infty)$. This proves (3.10). We finally define a measure $\bar{\mu}^\Gamma$ on $S^{\Gamma, \zeta^-} \times S^{\Gamma, \zeta^+}$ by

$$\bar{\mu}^\Gamma\{\sigma^1, \sigma^2)\} = \sum_{\substack{(\eta^1, \eta^2) \\ \eta^i \equiv \sigma^i \text{ on } \Gamma \setminus \Gamma(z)}} \mu^{\Gamma, \zeta^+, \zeta^-}\{(\eta^1, \eta^2)\} \, \mu^{\Gamma(z), \eta^1, \eta^2}\{(\sigma^1, \sigma^2)\}$$

Then $\bar{\mu}^\Gamma$ is a coupling of $\mu_{\beta, h_0}^{\Gamma, \zeta^\mp}$ and is above diagonal. We thus have that

$$\begin{aligned} F_z(l) &= \bar{\mu}^\Gamma\{\sigma_z^1 \neq \sigma_z^2\} \\ &= \sum_{(\eta^1, \eta^2)} \mu^{\Gamma, \zeta^+, \zeta^-}\{(\eta^1, \eta^2)\} \, \mu^{\Gamma(z), \eta^1, \eta^2}\{\sigma_z^1 \neq \sigma_z^2\} \end{aligned} \tag{3.12}$$

When $z \in Q_x(l/2)$, we can plug (3.10) in (3.12) to obtain

$$\begin{aligned} F_z(l) &\leqslant 2d(2(d+1)m)^{d-1} \exp\left(C_1 - \frac{m}{C_1}\right) \\ &\quad \times \max\{F_{z'}(l); z' \in \Gamma \cap Q_z((d+1)m+1)\} \end{aligned}$$

which proves (3.6) by taking m large enough.

4. PROOF OF PROPOSITION 2.4

In this section, we sketch the proof of Proposition 2.4, which is an extension of [Sch94a, Theorem 3]. Although the proof requires fairly long sequence of elaborate arguments and estimates as can be seen from that of [Sch94a], the same argument as in that paper works in the generality we need, with some modifications. So, we do not go further than to present some key lemmata, in which we indicate some essential changes we have to make to generalize the techniques in [Sch94a]. We begin by introducing some notations.

The set $\mathbf{B}$ of bonds in $\mathbf{Z}^d$ is defined by

$$\mathbf{B} = \{\{x, y\} \subset \mathbf{Z}^d; \|x - y\|_1 = 1\}$$

For a set Λ, we also define

$$\mathbf{B}_\Lambda = \{\{x, y\} \in \mathbf{B}; (x, y) \in \Lambda^2\}$$

$$\partial\Lambda = \{\{x, y\} \in \mathbf{B}; (x, y) \in \Lambda \times \Lambda^c\}$$

The number of *bonds* contained in a set $\gamma \subset \mathbf{B}$ will be denoted by $|\gamma|$.

A *contour* is a finite subset $\gamma \subset \mathbf{B}$ with the following properties; there exists a finite subset $\Theta \subset \mathbf{Z}^d$ such that both Θ and Θ^c is connected and that (ii) $\gamma = \partial\Theta$. The set Θ is uniquely determined by γ and hence is denoted by $\Theta(\gamma)$. Since a contour γ is a set of connected bonds in the sense explained in the bottom paragraph of [Sch94a, page 7], it follows that for each $b \in \mathbf{B}$ and $m = 1, 2,...,$

$$\sharp\{\gamma\text{: contour with } |\gamma| = m \text{ and } \gamma \ni b\} \leqslant \exp((m-1)\,\kappa_1) \tag{4.1}$$

for some $\kappa_1 = \kappa_1(d) \in (0, \infty)$ (See (4.24) in [Gri89], for example). For $1 \leqslant l$ and a finite set $S \subset \mathbf{B}$, let $\mathscr{F}_l(S)$ be the all possible choice of a family $\{\gamma_i\}_{i=1}^k$ $(k = 1,..., l)$ of contours such that $\sum_{i=1}^k |\gamma_i| = l$ and $\gamma_i \cap S \neq \phi$ for $i = 1,..., k$. Starting from (4.1), it is not difficult to prove that

$$\sharp\mathscr{F}_l(S) \leqslant \exp((|S| + l)\,\kappa_2) \tag{4.2}$$

for some $\kappa_2 = \kappa_2(d) \in (0, \infty)$ ([Sch94a, page 7–8]). If a contour γ is a subset of $\mathbf{B}_\Lambda \cup \partial\Lambda$ for some $\Lambda \subset \mathbf{Z}^d$, we say γ is a contour in Λ. For $\sigma \in S^\Lambda$, $\varepsilon = +$ or $-$ and $\Lambda \subset \mathbf{Z}^d$, a contour γ is said to be an (ε)-contour in Λ at σ if it satisfies

$$\partial_{\text{int}}\Theta(\gamma) \subset \{x \in \Lambda; \sigma(x) = \varepsilon 1\} \quad \text{and} \quad \partial_{\text{ext}}\Theta(\gamma) \subset \{x \in \Lambda; \sigma(x) = -\varepsilon 1\} \cup \Lambda(l)^c$$

A contour γ is said to be a contour in Λ at $\sigma \in S^\Lambda$ if it is either $(+)$-contour in Λ at σ or $(-)$-contour in Λ at σ. We take $E \subset \{\pm 1,..., \pm d\}$ such that (2.7) holds. For a contour γ, we define

$$|\gamma|_E = \sum_{\substack{1 \leqslant j \leqslant d \\ \{-j,\, j\} \subset E}} |\gamma^j| + \sum_{\substack{1 \leqslant j \leqslant d \\ \{-j,\, j\} \not\subset E}} \frac{|\gamma^j|}{2} \tag{4.3}$$

where $\gamma^j = \{b \in \gamma; b$ is parallel to x^j-axis.$\}$. We will need the following isoperimetric inequality;

$$|\gamma|_E \geqslant d 2^{(|E|/d)-1}\, |\Theta(\gamma)|^{(d-1)/d} \tag{4.4}$$

The constant $d2^{(|E|/d)-1}$ is sharp. In fact, (4.4) becomes equality for a rectangular contour γ of the form; $\Theta(\gamma) = Q(l; E)$, $(l = 1, 2,...,)$ we will need exactly the best constant in (4.4), rather than just the existence of some multiplicative constant for which an inequality of the form (4.4) is true.

Lemma 4.1. For a box Λ and contours $\{\gamma_i\}_{i=1}^k$ in Λ, such that $\Theta(\gamma_i) \cap \Theta(\gamma_j) = \phi$ if $i \neq j$, define

$$\mathscr{R}_{A,c,E} = \{\sigma \in S^A;\ |\Theta(\gamma)| \leqslant (c2^{(|E|/d)-1})^d \text{ for all contour in } A \text{ at } \sigma\} \tag{4.5}$$

$$S^A(\gamma_1,..., \gamma_k) = \{\sigma \in S^A;\ \{\gamma_i\}_{i=1}^k \text{ appears as } (+)\text{-contour in } A \text{ at } \sigma\} \tag{4.6}$$

Then

$$\mu_{\beta,h}^{A,(-;E)}(S^A(\gamma_1,..., \gamma_k) \mid \mathscr{R}_{A,c,E}) \leqslant \exp\left(-2\beta\left(1-\frac{ch}{d}\right), \sum_{i=1}^{k} |\gamma_i|_E\right) \tag{4.7}$$

Proof. We first prove the following energy gap estimate; For $\sigma \in \mathscr{R}_{A,c,E} \cap S^A(\gamma_1,..., \gamma_k)$

$$H^{A,(-;E)}(\sigma) - H^{A,(-;E)}(T\sigma) \geqslant 2\beta\left(1-\frac{ch}{d}\right)\sum_{i=1}^{k} |\gamma_i|_E \tag{4.8}$$

where $T\sigma \in S^A$ is defined by

$$T\sigma_x = \begin{cases} -\sigma_x, & \text{if } x \in \bigcup_{i=1}^k \Theta(\gamma_i) \\ \sigma_x & \text{if } x \notin \bigcup_{i=1}^k \Theta(\gamma_i) \end{cases} \tag{4.9}$$

Clearly,

$$H^{A,(-;E)}(\sigma) - H^{A,(-;E)}(T\sigma) \geqslant 2\beta \sum_{i=1}^{k} (|\gamma_i|_E - h\,|\Theta(\gamma_i)|) \tag{4.10}$$

On the other hand, by the definition of $\mathscr{R}_{A,c,E}$ and the isoperimetric inequality (4.4),

$$\begin{aligned} |\Theta(\gamma_i)| &= |\Theta(\gamma_i)|^{1/d} \cdot |\Theta(\gamma_i)|^{(d-1)/d} \\ &\leqslant c2^{(|E|/d)-1} \cdot \frac{2^{1-(|E|/d)}}{d}\,|\gamma_i|_E \\ &= \frac{c}{d}\,|\gamma_i|_E \end{aligned} \tag{4.11}$$

Plugging this into (4.10), we get (4.8). Once (4.8) is proved, the rest of the proof is the same as that of [Sch94a, Lemma 3], hence is omitted.

Lemma 4.2. Suppose that $\beta > \kappa_2/2$, where κ_2 is a constant in (4.2). Then for any $0 < A < d\,(1-\kappa_2/2\beta)$,

$$\begin{aligned} &\overline{\lim_{h \searrow 0}}\, h^{d-1} \log \mu^{Q(B/h;E),(-;E)}(\mathscr{R}_{Q(B/h;E),A/h,E}) \\ &\leqslant -2^{|E|-d}(m^{(+;E)}(\beta)\,B^d - dB^{d-1}) \end{aligned} \tag{4.12}$$

where

$$m^{(+;E)}(\beta) = \inf_{l \geqslant 1} \inf_{z \in Q(l;E)} \mu_{\beta,0}^{Q(l;E),(+;E)}(\sigma_z) \tag{4.13}$$

Proof. We have from the same argument as [Sch94a, Lemma 8] that

$$\begin{aligned} &\overline{\lim_{h \searrow 0}}\, h^{d-1} \log \mu_{\beta,h}^{Q(B/h;E),(-;E)}(\mathscr{R}_{Q(B/h;E),A/h,E}) \\ &\quad \leqslant \overline{\lim_{h \searrow 0}}\, h^{d-1} \left\{ m^{(+;E)}(\beta)\, h \left| Q\left(\frac{B}{h};E\right)\right| - \left|\partial Q\left(\frac{B}{h};E\right)\right|_E \right\} \end{aligned} \tag{4.14}$$

Since

$$\left| Q\left(\frac{B}{h};E\right)\right| \sim 2^{|E|-d}\left(\frac{B}{h}\right)^d$$

$$\left|\partial Q\left(\frac{B}{h};E\right)\right|_E \sim d\,2^{|E|-d}\left(\frac{B}{h}\right)^{d-1}$$

as $h \searrow 0$, the RHS of (4.14) is identified with that of (4.12).

Lemma 4.3. Suppose that two strictly monotone sequences $B_n \searrow d$ and $\beta_n \nearrow \infty$ satisfy $B_n m^{(+;E)}(\beta_n) > d$ for each $n = 1, 2,\ldots$ (cf. (4.13)). Then for each $\varepsilon > 0$, there is $n_0 \geqslant 1$ depending on the choice of ε, $\{B_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ such that for each $n \geqslant n_0$ and $\beta \geqslant \beta_n$

$$\overline{\lim_{h \searrow 0}}\, h^{d-1} \log \mu_{\beta,h}^{Q(B/h;E),(-;E)}\left(\left|\mathscr{C}\left(\frac{B_n}{h};E\right)\right| \geqslant \left(\frac{\varepsilon}{h}\right)^d\right) < 0 \tag{4.15}$$

where $\mathscr{C}(B_n/h;E)$ is the set of sites in $Q(B_n/h;E)$ which are connected to a site adjacent to $\bigcup_{j \in E} \partial_{\text{ext}}^{(j)} Q(B_n/h;E)$ by a $(-)$-path.

Proof. Since $d(1-\kappa_2/2\beta_n) \nearrow d \swarrow B_n$ as $n \nearrow \infty$, we can pick $n_0 \geqslant 1$ and $0 < A_n < d(1-\kappa_2/2\beta_n)$ such that

$$2^{|E|-d} B_n^d \leqslant 2^{|E|-d} A_n^d + \varepsilon^d$$

for $n \geqslant n_0$. Suppose that $|\mathscr{C}(B_n/h;E)(\sigma)| \geqslant (\varepsilon/h)^d$ and that $\gamma = \gamma(\sigma)$ is the $(+)$-contour in $Q(B_n/h;E)$ which maximizes $|\Theta(\gamma)|$. Then, $\sigma \in \mathscr{R}_{Q(B_n/h;E);A_n/h,E}$, since

$$|\Theta(\gamma)| \leqslant \left|Q\left(\frac{B_n}{h}, E\right)\right| - |\mathscr{C}(\varepsilon, E)(\sigma)|$$
$$\leqslant 2^{|E|-d}\left(\frac{B_n}{h}\right)^d - \left(\frac{\varepsilon}{h}\right)^d$$
$$\leqslant 2^{|E|-d}\left(\frac{A_n}{h}\right)^d$$

Therefore,

$$\mu_{\beta, h}^{Q(B_n/h;\, E),\, (-;\, E)}\left(\left|\mathscr{C}\left(\frac{B_n}{h}, E\right)\right| \geqslant \left(\frac{\varepsilon}{h}\right)^d\right) \leqslant \mu_{\beta, h}^{Q(B_n/h;\, E),\, (-;\, E)}(\mathscr{R}_{B_n/h,\, A_n/h,\, E})$$

(4.15) now follows from Lemma 4.2. Q.E.D.

The rest of the proof of Proposition 2.4 is sketched as follows. It is enough to consider the case $A < d$. In fact, the case $d \leqslant A$ can easily be reduced to the case of $A < d$ as follows. Define A', B' and h' by $A'/A = B'/B = h'/h = 2d/(A+B) \leqslant 1$ Then, $A' < d < B'$ and

$$\mu_{\beta, h}^{Q(B/h;\, E),\, (E;\, -)}\left(\mathscr{B}\left(\frac{A}{h}; E\right)\right) \leqslant \mu_{\beta, h'}^{Q(B'/h';\, E),\, (E;\, -)}\left(\mathscr{B}\left(\frac{A'}{h'}; E\right)\right)$$

by the FKG inequality. For $A < d$, we consider $\mu^{Q(B/h;\, E),\, (E;\, -)}$-probabilities of following two events separatedly;

$$\left\{\left|\mathscr{C}\left(\frac{d}{h}; E\right)\right| \geqslant \left(\frac{\varepsilon}{h}\right)^d\right\}, \qquad \mathscr{B}\left(\frac{A}{h}; E\right) \cap \left\{\left|\mathscr{C}\left(\frac{d}{h}; E\right)\right| \leqslant \left(\frac{\varepsilon}{h}\right)^d\right\}$$

By Lemma 4.3, the probability of the first event has an estimate of the form $C\exp(-C/h^{d-1})$ for large enough β. For the second probability, we can proceed in line with [Sch94a, Lemma 10, 11] to obtain an upper bound of the form $C\exp(-C/h)$ if ε is small and β is large. Putting these two together, we get Proposition 2.4.

5. PROOF OF PROPOSITION 2.5

We will use the following variational characterization of the spectral gap;

$$\mathrm{gap}_{\beta, h}^{\Lambda, \omega} = \inf\left\{\frac{-\mu^{\Lambda, \omega}(f A^{\Lambda, \omega} f)}{\mu^{\Lambda, \omega}(|f - \mu^{\Lambda, \omega} f|^2)};\ f \in \mathscr{C}_\Lambda\right\} \tag{5.1}$$

To describe an appropriate trial function $f \in \mathscr{C}_\Lambda$ to plug in, we make the following definitions. For $c > 0$ and a box Λ, define

$$\mathscr{R}_{\Lambda, c} = \{\sigma \in S^\Lambda;\ |\Theta(\gamma)| \leqslant (2c)^d \text{ for all } (+)\text{-contour in } \Lambda \text{ at } \sigma\} \tag{5.2}$$

$$\partial \mathscr{R}_{\Lambda, c} = \{\sigma \in \mathscr{R}_{\Lambda, c};\ \sigma^x \notin \mathscr{R}_{\Lambda, c} \text{ for some } x \in \Lambda\} \tag{5.3}$$

Now, by plugging the indicator function χ of the event $\mathscr{R}_{Q(b/h), A/h}$ $(0 < A < B)$ in (5.1), we obtain the following estimate of the spectral gap, which is reminiscient of the use of Cheeger constant in differential geometry.

$$\begin{aligned}
\mathrm{gap}_{\beta, h}^{Q(B/h), 0} &\leqslant \frac{-\mu^{Q(B/h), 0}(\chi A^{Q(B/h), 0} \chi)}{\mu^{Q(B/h), 0}(|\chi - \mu^{Q(B/h), 0} \chi|^2)} \\
&\leqslant \frac{C(d, \beta)}{\mu^{Q(B/h), 0}(\mathscr{R}_{Q(B/h), A/h})\, \mu^{Q(B/h), 0}(\mathscr{R}^{\mathrm{c}}_{Q(B/h), A/h})} \\
&\quad \times \sum_{x \in Q(B/h)} \sum_{\sigma \in \mathscr{R}_{Q(B/h), A/h}} \mu^{Q(B/h), 0}(\sigma)\, |\chi(\sigma^x) - 1| \\
&\leqslant \frac{C(d, \beta)\, |Q(B/h)|\, \mu^{Q(B/h), 0}(\partial \mathscr{R}_{Q(B/h), A/h} \mid \mathscr{R}_{Q(B/h), A/h})}{\mu^{Q(B/h), 0}(\mathscr{R}^{\mathrm{c}}_{Q(B/h), A/h})}
\end{aligned} \tag{5.4}$$

We will estimate the probabilities in the numerator and the denominator of (5.4) in a series of lemmata.

Lemma 5.1. For a box ${}_{Q(B/h), A/h}$ and contours $\{\gamma_i\}_{i=1}^k$ in Λ, such that $\Theta(\gamma_i) \cap \Theta(\gamma_j) = \phi$ if $i \neq j$, recall that we have defined an event $S^\Lambda(\gamma_1, \ldots, \gamma_k)$ by (4.6). Suppose now that $0 < A < d/(2(d+1))$. Then for any $B \geqslant 1$ and $h \in (0, 1)$

$$\mu_{\beta, h}^{Q(B/h), 0}(S^{Q(B/h)}(\gamma_1, \ldots, \gamma_k) \mid \mathscr{R}_{Q(B/h), A/h}) \leqslant \exp\left\{-\beta\left(1 - \frac{2A(d+1)}{d}\right) \sum_{i=1}^k |\gamma_i|\right\} \tag{5.5}$$

Proof. We first prove the following; for $\sigma \in \mathscr{R}_{Q(B/h), A/h} \cap S^{Q(B/h)}(\gamma_1, \ldots, \gamma_k)$

$$H^{Q(B/h), 0}(\sigma) - H^{Q(B/h), 0}(T\sigma) \geqslant \beta\left(1 - \frac{2A(d+1)}{d}\right) \sum_{i=1}^k |\gamma_i| \tag{5.6}$$

where $T\sigma \in S^\Lambda$ is defined by (4.9). Clearly,

$$H^{\Lambda, 0}(\sigma) - H^{\Lambda, 0}(T\sigma) \geqslant 2\beta \sum_{i=1}^k (|\gamma_i \cap \mathbf{B}_\Lambda|_E - h|\, \Theta(\gamma_i)|) \tag{5.7}$$

Since $|\Theta(\gamma_i)| \leqslant A^d |Q(B/h)|$, we see from (a careful reading of the proof of) [T89, Lemma A.1] that

$$\left|\gamma_i \cap \partial Q\left(\frac{B}{h}\right)\right| \leqslant \left(\frac{1}{2}+A\right)|\gamma_i| \tag{5.8}$$

On the other hand, we have from $|\Theta(\gamma_i)| \leqslant (2A/h)^d$ that $|\Theta(\gamma_i)| \leqslant A/dh\,|\gamma_i|$, (cf. the proof of (4.11)). Plugging this and (5.8) into (5.7), we get (5.6). Once (5.6) is proved, the rest of the proof is the same as that of [Sch94a, Lemma 3], hence is omitted. Q.E.D.

Lemma 5.2. Suppose that $0<A<d/(4(d+1))$ and that $\beta(\frac{1}{2}-2A(d+1)/d)>\kappa_2$, where κ_2 is the constant in (4.2). Then, there is $C(d,\beta)\in(0,\infty)$ such that for any $B\geqslant 1$ and $h\in(0,1)$,

$$\mu_{\beta,h}^{Q(B/h),0}(\partial\mathscr{R}_{Q(B/h),A/h} \mid \mathscr{R}_{Q(B/h),A/h}) \leqslant C(d,\beta)\left(\frac{2B}{h}\right)^{d-1}\exp\left\{-\beta d\left(\frac{2A}{h}\right)^{d-1}\right\} \tag{5.9}$$

Proof. We begin by observing how a configuration in $\partial\mathscr{R}_{A,c}$ looks like. For $\sigma\in\mathscr{R}_{A,c}$, $\sigma^x\notin\mathscr{R}_{A,c}$ is possible only when the following occurs; there is a collection of (+)-contours $\{\gamma_i\}_{i=1}^k$ in Λ at σ for some $k=1,2,\ldots$ such that $\gamma_i\ni x$ $(1\leqslant i\leqslant k)$ and such that the flipping of the spin at x makes them coalesce into a new (+)-contour γ in Λ at σ^x with $|\Theta(\gamma)|>(2c)^d$. We then have $\Theta(\gamma)=\{x\}\cup(\bigcup_{i=1}^k\Theta(\gamma_i))$, and thus

$$\sum_{i=1}^k|\Theta(\gamma_i)|\geqslant(2c)^d-1 \tag{5.10}$$

Since $|\Theta(\gamma_i)|\leqslant(2c)^d$ for $1\leqslant i\leqslant k$, we have that $|\Theta(\gamma_i)|\leqslant c\,|\gamma_i|/d$ (cf. the proof of (4.11)). Combining this and (5.10), we obtain

$$\sum_{i=1}^k|\gamma_i|\geqslant\frac{d}{c}((2c)^d-1) \tag{5.11}$$

From what we have observed above, we see

$$\mu_{\beta,h}^{Q(B/h),0}(\partial\mathscr{R}_{Q(B/h),A/h}\mid\mathscr{R}_{Q(B/h),A/h})$$
$$\leqslant\sum_{x\in Q(B/h)}\sum_{l\geqslant l_0(A/h)}\sum_{\{\gamma_i\}\in\mathscr{F}_l(x)}\mu_{\beta,h}^{Q(B/h),0}(S^{Q(B/h)}(\{\gamma_i\})\mid\mathscr{R}_{Q(B/h),A/h}) \tag{5.12}$$

where $l_0(A/h) = 2d\{(2A/h)^{d-1} - 1/2A\}$ and $\mathscr{F}_l(x)$ stands for all possible choice of a family $\{\gamma_i\}$ of contours in $Q(B/h)$ such that $\sum_{i=1}^{k} |\gamma_i| = l$ and $\gamma_i \ni x$. By (4.2), Lemma 5.1 and our assumption on A and β,

$$\begin{aligned}\sum_{\{\gamma_i\} \in \mathscr{F}_l(x)} & \mu_{\beta, h}^{Q(B/h), 0}(S^{Q(B/h)}(\{\gamma_i\}) \mid \mathscr{R}_{Q(B/h), A/h}) \\ & \leqslant \exp\left\{\kappa_2(l+1) - \beta\left(1 - \frac{2A(d+1)}{d}\right) l\right\} \\ & \leqslant \exp\left(\kappa_2 - \frac{\beta l}{2}\right)\end{aligned}$$

From this and (5.12), it is straightforward to conclude (5.9).

Lemma 5.3. For any $\varepsilon \in (0, 1)$ and $0 < A < d/2(d+1)$, there is $\beta_0(d, \varepsilon, A) \in (0, \infty)$ such that for $\beta \geqslant \beta_0$

$$\sup_{\substack{B \geqslant 1 \\ h \in (0, 1)}} \mu_{\beta, h}^{Q(B/h), 0}(\mathscr{R}_{Q(B/h), A/h}) \leqslant \frac{1+\varepsilon}{2} \tag{5.13}$$

Proof. We start with the following obvious inequality;

$$\mu_{\beta, h}^{Q(B/h), 0}(\mathscr{R}_{Q(B/h), A/h}) \leqslant \mu_{\beta, h}^{Q(B/h), 0}\{\sigma_0 = -1\} + \mu_{\beta, h}^{Q(B/h), 0}(\{\sigma_0 = 1\} \cap \mathscr{R}_{Q(B/h), A/h}) \tag{5.14}$$

The first probability in the RHS of (5.14) is less than 1/2 since $h > 0$. In a configuration of $\mathscr{R}_{Q(B/h), A/h} \cap \{\sigma_0 = 1\}$, the origin must be surrounded by a $(+)$-contour γ in $Q(B/h)$. Since a necessarily crosses the positive half of the x^1-axis, we have from Lemma 5.1 and (4.1) that

$$\begin{aligned}& \mu_{\beta, h}^{Q(B/h), 0}(\{\sigma_0 = 1\} \cap \mathscr{R}_{Q(B/h), A/h}) \\ & \leqslant \sum_{m=1}^{\infty} \sum_{s \geqslant m} \sum_{\substack{\gamma \\ \gamma \ni (m, 0, \ldots, 0),\, |\gamma| = s}} \mu_{\beta, h}^{Q(B/h), 0}(S^{Q(B/h)}(\gamma) \cap \mathscr{R}_{Q(B/h), A/h}) \\ & \leqslant \sum_{m=1}^{\infty} \sum_{s \geqslant m} \sum_{\substack{\gamma \\ \gamma \ni (m, 0, \ldots, 0),\, |\gamma| = s}} \exp\left\{-\beta\left(1 - \frac{2A(d+1)}{d}\right) s\right\} \\ & \leqslant \sum_{m=1}^{\infty} \sum_{s \geqslant m} \exp\left\{s\kappa_1 - \beta\left(1 - \frac{2A(d+1)}{d}\right) s\right\}\end{aligned}$$

which converges to 0 as $\beta \nearrow \infty$, uniformly in B and h. This completes the proof of the lemma. Q.E.D.

We now conclude the proof of Proposition 2.5. We take $A = d/(8(d+1))$. Then there is $\beta_0(d) \in (0, \infty)$ such that (5.9) holds for $\beta \geqslant \beta_0$. On the other hand, we see from Lemma 5.3 that, there is $\beta_1(d) \in (0, \infty)$ such that for $\beta \geqslant \beta_1$

$$\inf_{\substack{B \geqslant 1 \\ h \in (0, 1)}} \mu_{\beta, h}^{Q(B/h), 0}(\mathscr{R}^{\mathrm{c}}_{Q(B/h), A/h}) \geqslant \tfrac{1}{3} \tag{5.15}$$

Therefore, for $\beta \geqslant \max\{\beta_0, \beta_1\}$, we may plug (5.9) and (5.15) into (5.4) to get (2.10).

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